1 (a) $y' = e^{2x} + \sin(x)$

(i) ODE

(ii) order 1

(iii) linear since of the form

$$\underbrace{a_1(x)}_{1} y' + \underbrace{a_0(x)}_{0 \text{ in this case}} y = \overbrace{b(x)}^{e^{2x}+\sin(x)}$$

---

1 (b) $y'' - 4y = 0$

(i) ODE

(ii) order 2

(iii) linear since of the form

$$\underbrace{a_2(x)}_{1} y'' + \underbrace{a_1(x)}_{0} y' + \underbrace{a_0(x)}_{-4} y = \underbrace{b(x)}_{0}$$

$\boxed{①(c)}$ $y''' + 3y'' + 4y' + 12y = x^2 + x - 1$

(i) ODE

(ii) order 3

(iii) linear since of the form

$$\underbrace{a_3(x)}_{3}y''' + \underbrace{a_2(x)}_{3}y'' + \underbrace{a_1(x)}_{4}y' + \underbrace{a_0(x)}_{12}y = \underbrace{b(x)}_{x^2+x-1}$$

$\boxed{①(d)}$ $x^2y''' - 5y'' + \sin(x)y' - 2y = \cos(x) - 2$

(i) ODE

(ii) order 3

(iii) linear since of the form

$$\underbrace{a_3(x)}_{x^2}y''' + \underbrace{a_2(x)}_{-5}y'' + \underbrace{a_1(x)}_{\sin(x)}y' + \underbrace{a_0(x)}_{-2}y = \underbrace{b(x)}_{\cos(x)-2}$$

$\boxed{① \text{(e)}}$ $$\frac{d^2y}{dx^2} + yx^3\frac{dy}{dx} + x^2y = 0$$

(i) ODE

(ii) order 2

(iii) not linear because $\underbrace{\text{this coefficient}}$ has a y in it.

$$\frac{d^2y}{dx^2} + \boxed{yx^3}\frac{dy}{dx} + x^2y = 0$$

$\boxed{① \text{(f)}}$ $$\sin(x^2)\,y' + y = x$$

(i) ODE

(ii) order 1

(iii) linear since of the form

$$\underbrace{a_1(x)}_{\sin(x^2)}\,y' + \underbrace{a_0(x)}_{1}\,y = \underbrace{b(x)}_{x}$$

$\boxed{1(g)}$ $(2xy - y^3) + e^x \frac{dy}{dx} = 0$

(i) ODE

(ii) order 1

(iii) not linear because of the y^3 term

$\boxed{1(h)}$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$

(i) PDE

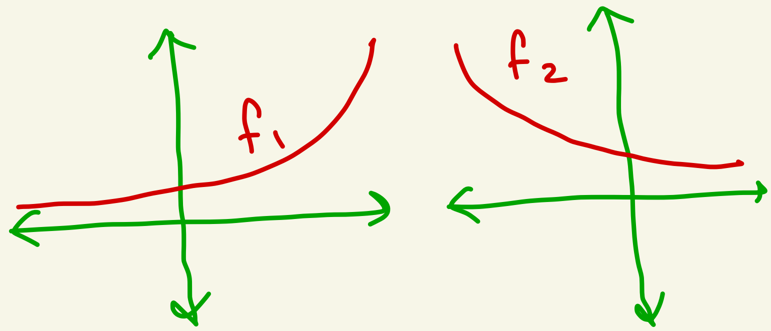
(ii) order 2 (highest order derivative is 2)

(iii) not applicable since not an ODE

$\boxed{(2)\,(a)}$ First note that $f_1(x) = e^{2x}$ and $f_2(x) = e^{-2x}$ are both defined on $I = (-\infty, \infty)$.

We have

$$\left.\begin{aligned} f_1(x) &= e^{2x} \\ f_1'(x) &= 2e^{2x} \\ f_1''(x) &= 4e^{2x} \end{aligned}\right\} \begin{array}{c}\text{all} \\ \text{defined} \\ \text{on} \\ I = (-\infty, \infty)\end{array}$$



Thus,

$$f_1'' - 4f_1 = 4e^{2x} - 4e^{2x} = 0$$

So, f_1 satisfies $y'' - 4y = 0$

Also,

$$\left.\begin{aligned} f_2(x) &= e^{-2x} \\ f_2'(x) &= -2e^{2x} \\ f_2''(x) &= -4e^{2x} \end{aligned}\right\} \begin{array}{c}\text{all} \\ \text{defined} \\ \text{on} \\ I = (-\infty, \infty)\end{array}$$

Thus,

$$f_2'' - 4f_2 = -4e^{2x} - 4e^{-2x} = 0$$

So, f_2 satisfies $y'' - 4y = 0$.

② (b)

We know from (a) that f_1 satisfies $y'' - 4y = 0$.

We also have that

$$f_1'(0) = 2e^{2(0)} = 2e^0 = 2$$

$$f_1(0) = e^{2(0)} = e^0 = 1$$

Thus, f_1 satisfies

$$y'' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

---

② (c)

We know from part (a) that f_2 solves $y'' - 4y = 0$.

We also have that

$$f_2'(1) = -2e^{-2(1)} = -2e^{-2}$$

$$f_2(1) = e^{-2(1)} = e^{-2}$$

Thus, f_2 satisfies

$$y'' - 4y = 0, \quad y(1) = e^{-2}, \quad y'(1) = -2e^{-2}$$

② (d)

Let $f(x) = c_1 f_1(x) + c_2 f_2(x) = c_1 e^{2x} + c_2 e^{-2x}$.

Then,

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$f''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x}$$

Thus,

$$f'' - 4f = 4c_1 e^{2x} + 4c_2 e^{-2x} - 4\left(c_1 e^{2x} + c_2 e^{-2x}\right) = 0$$

So, f satisfies $y'' - 4y = 0$

---

② (e) We know from part (d) that

$f(x) = c_1 e^{2x} + c_2 e^{-2x}$ satisfies $y'' - 4y = 0$.

We want

$$0 = f'(0) = 2c_1 e^{2(0)} - 2c_2 e^{-2(0)} = 2c_1 - 2c_2$$

$$1 = f(0) = c_1 e^{2(0)} + c_2 e^{2(0)} = c_1 + c_2$$

That is we need to solve

$$\boxed{\begin{aligned} 2c_1 - 2c_2 &= 0 \quad ① \\ c_1 + c_2 &= 1 \quad ② \end{aligned}}$$

Which is equivalent to

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \quad (1) \\ c_1 + c_2 &= 1 \quad (2) \end{aligned}}$$

(1) gives $c_1 = c_2$.

Plug this into (2) to get $c_2 + c_2 = 1$.

So, $c_2 = \frac{1}{2}$.

Thus, $c_1 = c_2 = \frac{1}{2}$.

Thus,

$$f(x) = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

satisfies

$$y'' - 4y = 0, \quad y'(0) = 0, \quad y(0) = 1$$

$\boxed{3}$ Let $\varphi(x) = 2\sqrt{x} - \sqrt{x}\ln(x)$

Note that φ is defined for all $x > 0$ that is on $I = (0, \infty)$.

$\boxed{3(a)}$ We have

$$\left.\begin{aligned}
\varphi(x) &= 2x^{1/2} - x^{1/2} \cdot \ln(x) \\
\varphi'(x) &= x^{-1/2} - \tfrac{1}{2} x^{-1/2} \cdot \ln(x) - x^{1/2} \cdot \tfrac{1}{x} \\
&= x^{-1/2} - \tfrac{1}{2} x^{-1/2} \cdot \ln(x) - x^{-1/2} \\
&= -\tfrac{1}{2} x^{-1/2} \cdot \ln(x) \\
\varphi''(x) &= \tfrac{1}{4} x^{-3/2} \cdot \ln(x) - \tfrac{1}{2} x^{-1/2} \cdot \tfrac{1}{x} \\
&= -\tfrac{1}{2} x^{-3/2} + \tfrac{1}{4} x^{-3/2} \cdot \ln(x)
\end{aligned}\right\} \begin{array}{l} \text{all} \\ \text{defined} \\ \text{on} \\ I = (0, \infty) \end{array}$$

Thus,

$$\begin{aligned}
4x^2 \cdot \varphi'' + \varphi &= 4x^2 \left[-\tfrac{1}{2} x^{-3/2} + \tfrac{1}{4} x^{-3/2} \cdot \ln(x) \right] \\
&\quad + 2x^{1/2} - x^{1/2} \cdot \ln(x) \\
&= -2x^{1/2} + x^{1/2} \cdot \ln(x) \\
&\quad + 2x^{1/2} - x^{1/2} \cdot \ln(x) = 0
\end{aligned}$$

Therefore, $\varphi(x) = 2\sqrt{x} - \sqrt{x}\ln(x)$
satisfies $4x^2y'' + y = 0$ on $I = (0, \infty)$.

$\boxed{3(b)}$ We also have that

$$\varphi(1) = 2\sqrt{1} - \sqrt{1} \cdot \underbrace{\ln(1)}_{0} = 2$$

$$\varphi'(1) = -\frac{1}{2}(1)^{-1/2} \cdot \underbrace{\ln(1)}_{0} = 0$$

Thus, from (a) and the above we know that φ solves the initial-value problem

$$4x^2y'' + y = 0, \quad y'(1) = 0, \quad y(1) = 2$$

④ Let $f_1(x) = e^{-3x}$, $f_2(x) = \cos(2x)$, $f_3(x) = \sin(2x)$

Note that all three functions are defined on $I = (-\infty, \infty)$.

Let's start with $f_1(x) = e^{-3x}$

We have

$$\left.\begin{aligned} f_1(x) &= e^{-3x} \\ f_1'(x) &= -3e^{-3x} \\ f_1''(x) &= 9e^{-3x} \\ f_1'''(x) &= -27e^{-3x} \end{aligned}\right\}$$

all defined on $I = (-\infty, \infty)$

So,

$$\begin{aligned} f_1''' + 3f_1'' + 4f_1' + 12f_1 &= -27e^{-3x} + 3\left[9e^{-3x}\right] + 4\left[-3e^{-3x}\right] + 12\left[e^{-3x}\right] \\ &= -27e^{-3x} + 27e^{-3x} - 12e^{-3x} + 12e^{-3x} \\ &= 0 \end{aligned}$$

Thus, $f_1(x) = e^{-3x}$ satisfies

$$y''' + 3y'' + 4y' + 12y = 0$$

$\boxed{\text{Now let's look at } f_2(x) = \cos(2x)}$

We have

$$\left.\begin{aligned} f_2(x) &= \cos(2x) \\ f_2'(x) &= -2\sin(2x) \\ f_2''(x) &= -4\cos(2x) \\ f_2'''(x) &= 8\sin(2x) \end{aligned}\right\} \text{ all defined on } I = (-\infty, \infty)$$

And,

$$\begin{aligned} &f_2''' + 3f_2'' + 4f_2' + 12f_2 \\ &\quad = 8\sin(2x) + 3(-4\cos(2x)) \\ &\qquad + 4(-2\sin(2x)) + 12\cos(2x) \\ &\quad = 8\sin(2x) - 12\cos(2x) \\ &\qquad - 8\sin(2x) + 12\cos(2x) \\ &\quad = 0 \end{aligned}$$

Thus, $f_2(x) = \cos(2x)$ satisfies

$$y''' + 3y'' + 4y' + 12y = 0$$

$\boxed{\text{Now let's look at } f_2(x) = \sin(2x)}$

We have

$$\left.\begin{aligned} f_2(x) &= \sin(2x) \\ f_2'(x) &= 2\cos(2x) \\ f_2''(x) &= -4\sin(2x) \\ f_2'''(x) &= -8\cos(2x) \end{aligned}\right\} \text{ all defined on } I = (-\infty, \infty)$$

And,

$$\begin{aligned} &f_2''' + 3f_2'' + 4f_2' + 12f_2 \\ &\quad = -8\cos(2x) + 3(-4\sin(2x)) \\ &\qquad + 4(2\cos(2x)) + 12\sin(2x) \\ &\quad = -8\cos(2x) - 12\sin(2x) \\ &\qquad + 8\cos(2x) + 12\sin(2x) \\ &\quad = 0 \end{aligned}$$

Thus, $f_2(x) = \sin(2x)$ satisfies

$$y''' + 3y'' + 4y' + 12y = 0$$